ISRAEL JOURNAL OF MATHEMATICS, Vol. 65, No. 3, 1989

COMPUTING LANNES T FUNCTOR

BY
J. AGUADÉ
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain

ABSTRACT
We compute the value of the functor T of Lannes on a certain class of algebras over the Steenrod algebra which includes polynomial algebras.

1. For a fixed odd prime p, denote by $\mathcal{K}$ and $\mathcal{U}$ the categories of unstable algebras and unstable modules over the mod p Steenrod algebra. Denote by $H^*(-)$ the cohomology functor with coefficients in the field $\mathbf{F}_p$. Let V be an n-dimensional vector space over $\mathbf{F}_p$. Lannes has introduced a functor T_V in the categories $\mathcal{K}$ and $\mathcal{U}$ which is left adjoint to the functor $-\otimes H^*(BV)$, i.e., one has natural isomorphisms

$$\mathrm{Hom}(M, N \otimes H^*(BV)) \cong \mathrm{Hom}(T_V(M), N),$$

where Hom is taken in the appropriate category. On the category $\mathcal{U}$, T_V is an exact functor which commutes with tensor products ([3]). This implies that, for an algebra M in $\mathcal{K}$, the value of the functor T_V on the underlying module of M is the underlying module of the algebra $T_V(M)$. If $n = 1$, we write T instead of T_V. It is clear that T_V coincides with the n-iterated composition $T \circ \cdots \circ T$.

The usefulness of these functors comes from the fact that in many important cases they can be used to compute the cohomology of the mapping space $\mathrm{map}(BV, X)$. If X is a p-complete space such that $H^*(X)$ is of finite type, Lannes shows that the vanishing of $T_V(H^*(X))$ in degree 1 is a sufficient condition for the existence of an isomorphism

$$T_V(H^*(X)) \cong H^*(\mathrm{map}(BV, X))$$

induced by the adjoint of the evaluation map. If $f\colon BV \to X$ is a map, then from the above isomorphism one can easily compute the cohomology of the

Received May 2, 1988

component of map(BV, X) containing f. Let us consider the homomorphism $\omega: T_V(H^*(X)) \to \mathbf{F}_p$ which is adjoint to the homomorphism induced by f in cohomology. If T^0 denotes the degree zero component of T_V, then $\mathbf{F}_p$ is a $T^0(H^*(X))$-module and we can consider

$$T_V^f(H^*(X)) = T_V(H^*(X)) \underset{T^0(H^*(X))}{\otimes} \mathbf{F}_p.$$

Then, $T_V^f(H^*(X))$ computes the cohomology of the space map$(BV, X)_f$.

During the Barcelona conference (April 1986) it became clear that Lannes theory can be used to provide alternative proofs of some of the results presented. In this note, which was written after the conference, we work out the computation of the functor T_V for an important class of unstable algebras over the Steenrod algebra. As an immediate consequence, we obtain some of the results which were announced by Wilkerson in his talk in the BCAT ([2]). In particular, we prove the separability of the Adams–Wilkerson embedding.

2. Let us denote by $P(r)$ the graded algebra $\mathbf{F}_p[x]$, $\deg x = 2p^r$, with the unstable action of the mod p Steenrod algebra given by the embedding

$$P(r) \to \mathbf{F}_p[t] = P(0) = H^*(BS^1)$$

given by $x \mapsto t^i$, $i = p^r$. Notice that the Steenrod operations P^i, $i = 1, \ldots, p^r - 1$, as well as the Bockstein homomorphism, vanish on the generator of $P(r)$. Let us denote by v and u the generators in degree 1 and 2 in $H^*(B\mathbf{Z}_p)$.

PROPOSITION 1. *Let $M \in \mathcal{K}$ and let $x \in M \otimes H^*(B\mathbf{Z}_p)$. Let us write x as*

$$x = \sum_{i \geq 0} a_i \otimes u^i + \sum_{j \geq 0} b_j \otimes u^j v. \tag{1}$$

Assume that $\beta x = P^1 x = \cdots = P^{p^r} x = 0$ and let N be the sub-A-module of M generated by a_k, $k \equiv 0$ (p^{r+1}). Then the coefficients a_i and b_i, $i \geq 0$, belong to N.

PROOF. By applying the Bockstein homomorphism to (1) one sees that $b_i = \pm \beta a_{i+1}$. Since the ideal of $M \otimes H^*(B\mathbf{Z}_p)$ generated by v is closed under the action of the Steenrod powers, we can assume, without loss of generality, that $b_i = 0$, $i \geq 0$.

We prove the proposition by induction on r. In the case $r = 0$ we have

$$0 = P^1 x = \sum_{i \geqq 0} P^1 a_i \otimes u^i + \sum_{i \geqq 0} i a_i \otimes u^{i+p-1}$$

and hence $ia_i = -P^1 a_{i+p-1}$. Let N_s be the sub-A-module of M generated by a_i, $i \geqq s$. For s big enough, the module N_s vanishes. Let N be the sub-A-module of M generated by a_k, $k \equiv 0\ (p)$. Assume $N_{s+1} \subseteq N$, $N_s \not\subseteq N$. We have $s \not\equiv 0\ (p)$ and then $a_s = P^1 a_{s+p-1}$ (modulo units) and $a_s \in N_{s+p-1} \subseteq N$. This proves $N_0 = N$ and the proposition in the case $r = 0$.

In the general case we have

$$0 = P^{p^r} x = \sum_{\substack{i \geqq 0 \\ s = 0, \ldots, p^r}} \binom{i}{s} P^{p^r - s} a_i \otimes u^{i + s(p-1)}.$$

For each i we get the identity

$$\binom{i}{p^r} a_i + \binom{i+p-1}{p^r - 1} P^1 a_{i+p-1} + \cdots + \binom{i + p^r(p-1)}{0} P^{p^r} a_{i + p^r(p-1)} = 0.$$

Notice that, for $i \equiv 0\ (p^r)$, $i \not\equiv 0\ (p^{r+1})$, we have

$$\binom{i}{p^r} \not\equiv 0\ (p).$$

Then the induction hypothesis and an argument similar to the one in the case $r = 0$ prove that a_i, $i \geqq 0$, belong to N. ∎

COROLLARY 2. *If $\phi : P(r) \to M \otimes H^*(B\mathbf{Z}_p)$ is an A-algebra homomorphism then there are elements a, b in M such that $\phi(x) = a \otimes 1 + b \otimes u^{p^r}$.*

PROOF. If we write $\phi(x)$ as

$$\phi(x) = \sum_{i=0}^{p^r} a_i \otimes u^i + \sum b_j \otimes u^j v,$$

Proposition 1 implies that the coefficients a_i, b_j belong to the sub-A-module of M generated by a_0 and a_{p^r}. Since a_{p^r} has degree zero and a_0 has maximal degree, we obtain that any other coefficient must vanish. ∎

This corollary determines the functor T on the algebra $P(r)$:

$$T(P(r)) \cong T^0(P(r)) \otimes P(r)$$

where T^0, the degree zero component of T, is given by

$$T^0(P(r)) \cong \mathbf{F}_p[\alpha]/(\alpha^p - \alpha),$$

the free p-boolean algebra on one generator. Recall now that T commutes with tensor products ([3]) and that $TM = M$ if M is concentrated in degree zero. If we denote by $P(r_1, \dots, r_n)$ the algebra $P(r_1) \otimes \cdots \otimes P(r_n)$, we obtain

$$T_V(P(r_1, \dots, r_n)) \cong T^0 \otimes P(r_1, \dots, r_n),$$

$$T^0 \cong \mathbf{F}_p[\alpha_{ij};\, 0 \leqq i, j \leqq n]/(\alpha_{ij}^p - \alpha_{ij}).$$

We will denote this algebra T^0 by $A(n)$. The adjointion homomorphism

$$\mathrm{Hom}_{\mathscr{K}}(A(n), \mathbf{F}_p) \cong \mathrm{Hom}_{\mathscr{K}}(P(r_1, \dots, r_n), H^*(BV))$$

can be described in the following way. If $f(\alpha_{ij}) = \lambda_{ij}$, then the adjoint of f is given by $f'(x_i) = \sum \lambda_{ij} u_j$.

3. Let G be a subgroup of $\mathrm{GL}_n\mathbf{F}_p$. G acts linearly on $P = P(0, \dots, 0) = \mathbf{F}_p[t_1, \dots, t_n]$. Since this action commutes with the action of the Steenrod operations, the subalgebra P^G of invariant elements inherits the structure of an unstable algebra over the mod p Steenrod algebra. If g is an element of G we denote by g_{ij} the entries in the matrix corresponding to g, in such a way that the action of g on a generator t_j is given by $g \cdot t_j = \sum g_{ij} t_i$. By naturality, the group G will also act on $T_V(P)$. The action of an element g on α_{ij} is given by $g \cdot \alpha_{ij} = \sum g_{ki} \alpha_{kj}$. This follows easily from the commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(A(n), \mathbf{F}_p) & \rightarrow & \mathrm{Hom}(P, H^*(BV)) \\ {\scriptstyle g^*}\uparrow & & {\scriptstyle g^*}\uparrow \\ \mathrm{Hom}(A(n), \mathbf{F}_p) & \rightarrow & \mathrm{Hom}(P, H^*(BV)) \end{array}$$

which gives the action of g on the duals ω_{ij} of the generators α_{ij}.

PROPOSITION 3. $T_V(P^G) \cong [A(n) \otimes P]^G$.

PROOF. If M is any G-module it is clear that

$$M^G = \bigcap_{g \in G} \ker(1 - g).$$

Hence, if G is finite then any additive exact functor commutes with the invariant submodule functor. ■

4. If G acts linearly on P we can consider the homomorphisms

$$P^G \to P \to H^*(BV)$$

where the second homomorphism is the one induced in cohomology by the canonical map $BV \to BT^n$. The composition homomorphism ϕ induces a homomorphism

$$\omega : T_V(P^G) \to \mathbf{F}_p.$$

We want to compute

$$T_V^\phi(P^G) = T_V(P^G) \underset{A(n)^G}{\otimes} \mathbf{F}_p.$$

First of all, notice that the action of the augmentation ω on the generators α_{ij} of $A(n)$ is given by $\omega(\alpha_{ij}) = \delta_{ij}$ because, as one sees easily, $\omega = \Sigma\, \omega_{ii}$ for ω_{ij}, the basis dual to α_{ij}.

We have a homomorphism

$$\psi : T_V^\phi(P^G) \to P$$

given by

$$T_V^\phi(P^G) = [A(n) \otimes P]^G \underset{A(n)^G}{\otimes} \mathbf{F}_p \to [A(n) \otimes P] \underset{A(n)}{\otimes} \mathbf{F}_p \cong P.$$

PROPOSITION 4. *ψ is an isomorphism.*

PROOF. If A is a graded $\mathbf{F}_p$-algebra such that A^0 is a p-boolean ring (i.e. $x^p = x$ for all x in A^0) with finitely many maximal ideals $I_1, \ldots, I_n$, then there is a natural decomposition of A as a product of connected graded algebras

$$A \xrightarrow{\cong} (A/I_1A) \times \cdots \times (A/I_nA).$$

This decomposition corresponds to the decomposition of a space as a disjoint union of components and follows easily from the fact that, since A^0 is p-boolean, the residue field is always $\mathbf{F}_p$ and any non-zero element is avoided by some maximal ideal.

This decomposition is natural in the following sense. If $A \to B$ is an algebra homomorphism and I is a maximal ideal of A^0, then we have an homomorphism $A/IA \to B/JB$ for any $J \in \operatorname{Spec} B^0$ which lies over I. Assume A is a subalgebra of B. In this case, we easily see that A/IA injects on B/JB if, for instance, there is only one point of Spec B^0 over I. More in general, we get an injection if

$$A \cap JB \subset \bigcap_{J' \cap A = I} J'B.$$

When we apply this decomposition to our case, we see that $T^{\phi}_V(P^G)$ and $T^{\phi}_V(P)$ are the direct factors of $T_V(P^G)$ and $T_V(P)$ corresponding to the maximal ideal $\ker \omega$. The spectra of the degree zero components can be identified with $\mathrm{Hom}_{\mathscr{K}}(P^G, H^*(BV))$ and $\mathrm{Hom}_{\mathscr{K}}(P, H^*(BV))$, respectively. Hence, one sees that the fiber over $I = \ker \omega$ is formed by the ideals gJ, for $g \in G$ and J any point of the fiber. This shows that the above condition for a monomorphism is satisfied and so the inclusion $P^G \subset P$ induces a monomorphism $T^{\phi}_V(P^G) \to T^{\phi}_V(P)$.

To complete the proof of the proposition we will construct a section homomorphism

$$\phi : P \to [A(n) \otimes P]^G \bigotimes_{A(n)^G} \mathbf{F}_p.$$

We use the following universal construction. If $X = (x_{ij})$ is a matrix whose entries are indeterminates, then there is a matrix $Y = (y_{ij})$ whose entries are integral polynomials on the x_{ij} and such that $XY = (\det X)I$ where I denotes the identity matrix. We define the matrix $\bar{X} = (\bar{x}_{ij})$ as

$$\bar{X} = Y(XY)^{p-2} = (\det X)^{p-2} Y.$$

The entries of $\bar{X}$ are integral polynomials on the indeterminates x_{ij} and so this universal construction can be done in any commutative ring. Notice that $\overline{XY} = \bar{Y}\bar{X}$. We obtain matrices $(\bar{\alpha}_{ij})$ and $(\bar{g}_{ij})$. Notice that $(\bar{g}_{ij})$ is the inverse matrix of (g_{ij}). It is not difficult to compute the action of an element G of g on the elements $\bar{\alpha}_{ij}$ of the algebra $A(n)$. We have

$$(\bar{\alpha}_{ij})(\bar{g}_{ij}) = \overline{(g_{ij})(\alpha_{ij})} = \overline{(g^{t}, \alpha_{ij})} = (g^{t} \cdot \bar{\alpha}_{ij}),$$

and so

$$g \cdot \bar{\alpha}_{ij} = \sum \bar{g}_{jk} \bar{\alpha}_{ik}.$$

Let us consider, for $i = 1, \ldots, n$, the element

$$w_i = \sum \bar{\alpha}_{ij} t_j \in A(n) \otimes P.$$

For any g in G we have

$$g \cdot w_i = \sum_j (g \cdot \bar{\alpha}_{ij})(g \cdot t_j) = \sum_{j,k,s} \bar{g}_{jk} \bar{\alpha}_{ik} g_{sj} t_s = w_i.$$

Hence, the elements w_i belong to the invariant submodule $[A(n)\otimes P]^G$. This allows us to define

$$\phi: P \to [A(n)\otimes P]^G \bigotimes_{A(n)^G} \mathbf{F}_p,$$

$$\phi(t_i) = w_i \otimes 1.$$

Notice that ϕ is a homomorphism of unstable algebras over the mod p Steenrod algebra. It remains only to check that $\psi\phi = 1$. This follows from $\omega(\bar{\alpha}_{ij}) = \delta_{ij}$ which is an immediate consequence of the definition of $(\bar{\alpha}_{ij})$. ■

5. The theorems of Adams–Wilkerson ([1]) and Wilkerson ([4]) show that the above computation determines the value of the functor T_V on a wide category of unstable algebras over the mod p Steenrod algebra. According to Wilkerson ([4]) if A in $\mathcal{K}$ is a connected integral domain of transcendence degree n and if moreover A is noetherian and integrally closed, then there is a group G in $\mathrm{GL}_n\mathbf{F}_p$ and there are integers $0 \leqq r_1 \leqq \cdots \leqq r_n$, such that

$$A = P^G \cap P(r_1, \ldots, r_n).$$

Hence:

PROPOSITION 5. *If* $A \in \mathcal{K}$ *satisfies the above conditions and* $\phi: A \to H^*(BV)$ *is induced by the Adams–Wilkerson embedding, then* $T_V^\phi A \cong P(r_1, \ldots, r_n)$ *for some* $0 \leqq r_1 \leqq \cdots \leqq r_n$.

PROOF. Notice that we can assume that G acts on $P(r_1, \ldots, r_n)$ and A consists of the elements of $P(r_1, \ldots, r_n)$ which are invariant under the action of G. We have

$$T_V(A) = T_V(P^G) \cap T_V(P(r_1, \ldots, r_n))$$

and we want to see that the same is true with T_V^ϕ. Write

$$A = (A(n)\otimes P)^G, \quad B = A(n)\otimes P(r_1, \ldots, r_n), \quad C = A(n)\otimes P.$$

Let J be the ideal of $A(n)$ given by ω and let $I = A(n)^G \cap J$. Then, as we have already discussed, we have inclusions

$$(A/IA) \to (C/JC), \quad (B/JB) \to (C/JC), \quad (A \cap B)/I(A \cap B) \to A/IA$$

and hence we have an inclusion

$$j: A \cap B/I(A \cap B) \to A/IA \cap B/JB.$$

We want to prove that this is an isomorphism. Let $a \in A$, $b \in B$ such that $a \equiv b$ (JC). We can assume that $a \equiv 0$ $(I_i A)$ for any $I_i \neq I$ and $b \equiv 0$ $(J_j B)$ for any J_j not over I. Recall that the fiber over I is formed by the ideals gJ, $g \in G$. Since a is invariant under G, we have $a \equiv gb$ (gJC). Hence, $a \in A \cap B$ and j is an isomorphism. ■

Lannes theory and the non-realizability of the algebras $P(r_1, \ldots, r_n)$ for $r_n \neq 0$ as cohomology rings provide a proof of the following important result.

THEOREM 6 (Dwyer–Miller–Wilkerson [2]). *If A is a connected unstable algebra over the mod p Steenrod algebra (p odd) which is an integral domain of finite transcendence degree and which is noetherian and integrally closed, then the separability of the Adams–Wilkerson embedding is a necessary condition for A to be realizable as a cohomology ring. In other words, if A is realizable, then A is an algebra of invariants.* ■

REFERENCES

1. J. F. Adams and C. Wilkerson, *Finite H-spaces and algebras over the Steenrod algebra*, Ann. Math. **111** (1980), 95–143.

2. W. Dwyer, H. Miller and C. Wilkerson, *Uniqueness of classifying spaces*, notes distributed in the 1986 BCAT.

3. J. Lannes, *Sur la cohomologie modulo p des p-groupes abeliens elementaires*, preprint.

4. C. Wilkerson, *Rings of invariants and inseparable forms of algebras over the Steenrod algebra*, preprint.